

Notes on Engel groups and Engel elements in groups. Some generalizations

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Abstract

Engel groups and Engel elements became popular in 50s. We consider in the paper the more general nil-groups and nil-elements in groups. All these notions are related to nilpotent groups and nilpotent radicals in groups. These notions generate problems which are parallel to Burnside problems for periodic groups.

The first three theorems of the paper are devoted to nil-groups and Engel groups, while the other results are connected with the further generalizations. These generalizations extend the theory to solvable groups and solvable radicals in groups. The paper has two parts. The first one (sections 2-4) deals with old ideas, while the second one (sections 5-9) is devoted to generalizations.

1 Introduction

This paper is a nostalgic reminiscence on group theory of 50s (just last century). In some sense this feedback to the past is inspired by the paper [BGGKPP] and by the recent talks on *PI*-algebras by L. Rowen and A. Kanel-Belov. Recall some definitions and some necessary old results.

We distinguish Engel groups and nil-groups, Engel elements and nil-elements [Pl3].

Let $F_2 = F(x, y)$ be the free group. Define the sequence

$$e_1(x, y) = [x, y], e_2(x, y) = [e_1(x, y), y], \dots, e_n(x, y) = [e_{n-1}(x, y), y],$$

where $x, y \in F_2$. Let now G be an arbitrary group.

Definition 1.1. An element $g \in G$ is called nil-element if for every $a \in G$ there is $n = n(a, g)$ such that $e_n(a, g) = 1$.

Definition 1.2. A group G is called *nil-group* if every its element is nil-element.

Every locally-nilpotent group is a nil-group, but the opposite is not true ([GS]).

Definition 1.3. A group G is called *Engel group* if it satisfies an identity $e_n(x, y) \equiv 1$ for some n .

In this case we call the group n -Engel. The variety of n -Engel groups is denoted by E_n and let F_k^n be the free group with k free generators in this variety .

There is a long-standing conjecture that the group F_k^n is not nilpotent but up to now there are no reasonable approaches to this problem. We show (Theorem 1) the restricted solution of this problem similar to a solution of the restricted Burnside problem (see for example [Ne],[Ko], etc.)

Definition 1.4. An element $g \in G$ is called Engel element, if there exists $n = n(g)$ such that for every $x \in G$ the identity $e_n(x, g) \equiv 1$ holds in G .

Thus, the definition of an Engel element differs from the definition of a nil-element. However, following the tradition sometimes we use the term Engel element also for nil-elements (with the meaning unbounded Engel elements).

The conditions on a group G provided the set of all nil-elements or (and) the set of all Engel elements constitute a subgroup are considered. In view of the latter problem note the following general result [Pl2]:

Let the group G has an ascending normal series with the locally noetherian quotients. Then the set of all its nil-elements constitute the subgroup in G coinciding with the locally nilpotent radical $HP(G)$.

This Theorem has been preceded by the similar theorem for the case when the factors of the normal series are locally nilpotent [Pl1] and the theorem of Baer [Ba], stating that the nilpotent radical of a Noetherian group coincides with the set of its nil-elements.

Theorem of Baer follows from the Lemma [Pl2] we are going to recall:

Let G be an arbitrary group, g its nil-element. Then there exists in G a normal series of nilpotent subgroups

$$H_1 \subset H_2 \dots, H_n \subset \dots,$$

where $H_1 = \{g_1\}$, $H_n = \{H_{n-1}, h_n g h_n^{-1}\}$ for some $h_n \in G$. Here and elsewhere $\{ \}$ stands for the subgroup generated by some elements. This series stops on some place n if H_n is a normal subgroup in G .

Note here the following result of A. Tokarenko [To1], see also [Pl3]:

Let a group G be a subgroup in some $GL_n(K)$, where K is a commutative ring with 1. Then the set of all nil-elements in G is the locally nilpotent radical $HP(G)$.

Now we formulate two theorems of this paper:

Theorem 1. *In any variety E_n all its locally nilpotent groups form a subvariety.*

In the second theorem we consider PI -groups that is the groups which can be embedded to a group of invertible elements of some PI - algebra over a field P .

For example, the full matrix group $GL_n(P)$ and all its subgroups are PI -groups.

It has been proved by Procesi [Pr] and Tokarenko [To2] that every periodic PI -group is locally finite. The following theorem has the similar flavor:

Theorem 2. *Every nil- PI -group G is locally nilpotent.*

In fact we will prove the more general result:

Theorem 3. *In every PI -group G the set of all its nil-elements coincides with the locally nilpotent radical $HP(G)$.*

Since other results will require additional definitions, they will be formulated later.

Engel groups and Engel elements in groups are related to nilpotent groups and nilpotent radicals in groups. Along with these elements we consider also their Engel-like generalizations, which are tied with solvable groups and solvable radicals in groups.

Note also two facts which will be used in the sequel.

First of all this is the theorem by Wilson ([Wi]) which states that every residually finite finitely generated Engel group is nilpotent.

Second, we use the following Kaluzhnin's theorem [Ka]. Let the group G acts unitriangularly and faithfully in the space V . Then the group G is nilpotent. Unitriangularity means that there is a series

$$V = V_0 \supset V_1 \supset \dots \supset V_n = 0$$

in V such that all members of the series are invariant in respect to the action of the group G and in all quotients of the series the group G acts trivially. If the group G acts in V faithfully then G has the nilpotency class $n - 1$.

2 Proof of the theorem 1.

Prove first the following:

Proposition 2.1. *In the variety E_n for any natural k there exists a nilpotent group with k generators \tilde{F}_k^n such that every k -generated nilpotent group $G \in E_n$ is a homomorphic image of the group \tilde{F}_k^n .*

Proof. Let us start with the free in E_n group F_k^n . Let H_k^n be the intersection of all $H \triangleleft F_k^n$ with nilpotent F_k^n/H . The group $\tilde{F}_k^n = F_k^n/H_k^n$ is residually nilpotent. This group is also residually finite since every finitely generated nilpotent group is residually finite. Besides, this group is Engel. By the result from [Wi] the group \tilde{F}_k^n is nilpotent.

Let G be an arbitrary group in E_n with k generators. There is a surjection $F_k^n \rightarrow G$. Let H be the kernel of this homomorphism. Then $H \supset H_k^n$ and this gives a surjection $\tilde{F}_k^n \rightarrow G$. \square

This proposition is equivalent, in fact, to the theorem 1. Indeed, let Θ be the class of all locally nilpotent groups in E_n . This class is closed in respect to taking subgroups and homomorphic images. Let us check that the class is closed in respect to Cartesian products.

Let $G = \prod_{\alpha \in I} G_\alpha$ where all G_α are locally nilpotent groups in the variety E_n . Take a finitely generated subgroup H in G with k generators. The group H is approximated by k generated nilpotent groups $H_\alpha \subset G_\alpha$. The nilpotency class of all H_α is bounded by the nilpotency class of the group \tilde{F}_k^n . Thus, H is nilpotent and G is locally nilpotent.

This means that the class Θ is a variety. It is easy to see that the group \tilde{F}_k^n is the free group with k generators in the variety Θ .

3 PI-groups

Let us fix a field P and for every group G consider a representation $G \rightarrow A$, where A is an associative algebra with 1 over the ground field P and the arrow means a homomorphism of the group G to the group of invertible

elements of the algebra A . If this representation is faithful then we say that the algebra A is a linear envelope of the group G . The group algebra PG is the universal linear envelope. We consider the groups G from the point of view of the possible linear envelopes. In particular, G is a linear group if it has a finite dimensional linear envelope.

Definition 3.1. A group G is a *PI*-group if it has a linear envelope A which is a *PI*-algebra.

Let us fix A and let G_0 be the group of invertible elements of the algebra A . We consider G as a subgroup in G_0 .

In a *PI*-algebra A consider a series of ideals:

$$U_0 = 0 \subset U_1 \subset U_2 \subset A,$$

where U_1 is the sum of the nilpotent ideals of A and U_2 is the Levitzky radical of A . It is known [J] that U_2/U_1 is the nilpotent algebra and there is a faithful embedding $A/U_2 \rightarrow M_n(K)$. Here $M_n(K)$ is the matrix algebra of the dimension n and K is a commutative ring with 1 which is a cartesian sum of fields. The group of invertible elements of $M_n(K)$ is $GL_n(K)$.

Proposition 3.1. *Let G be a *PI*-group. Then there is a chain of normal subgroups*

$$1 = H_0 \subset H_1 \subset H_2 \subset G,$$

where H_1 is generated by the nilpotent normal subgroups in G , H_2 is locally nilpotent, and there is a faithful embedding

$$G/H_2 \rightarrow GL_n(K),$$

where $K = \bigoplus P_\alpha$, P_α is a field.

Proof. First recall the known things. Let A be an associative algebra with 1, G_0 the group of invertible elements in A , G a subgroup in G_0 . The group G acts in the space A by the rule: $a \rightarrow ag$, $a \in A$, $g \in G$.

Let U be a two-sided ideal in A and $\mu : A \rightarrow A/U$ the natural homomorphism. It induces the representation $\mu : G \rightarrow A/U$. Then $\mu_0(1)$ is the coset $1 + U$ and the kernel of μ_0 is the set of elements $g \in G$, such that $g - 1 \in U$, i.e., $g \in 1 + U$. We have $\text{Ker}(\mu_0) = G \cap (1 + U)$. The group G acts also in A/U with the same kernel $G \cap (1 + U)$.

Consider the coset $1 + U$ and let U be a locally nilpotent ideal. Check that $H = 1 + U$ is a locally nilpotent normal subgroup in G_0 . The set H is closed under multiplication. Let $a \in U$ and $a^n = 0$. We have $(1 + a)(1 - a + a^2 - \dots + (-1)^{n-1}a^{n-1}) = 1$ and $1 + a$ is an invertible element. Thus, $H = 1 + U$ is a subgroup in G_0 . This subgroup is normal since it coincides with the kernel $G_0 \rightarrow A/U$. It remains to check that H is locally nilpotent.

Consider first the case when U is a nilpotent ideal. Consider a series

$$U = U_0 \supset U_1 \supset \dots \supset U_k \supset \dots \supset U_n = 0,$$

where U_k is a linear combination of the elements of the form

$$a(g_1 - 1) \dots (g_k - 1), a \in U, g_i \in H.$$

The series above is invariant under the action of H and H acts trivially in the factors. Besides, H acts trivially in A/U . This means that H acts in A unitriangularly and faithfully. Thus, by Kaluzhnin's theorem H is nilpotent.

Let now U be a locally-nilpotent ideal, $H_0 = \{g_1, \dots, g_n\}$ be a finitely generated subgroup in H . Assume that for every generator g_i its inverse belongs to the set $\{g_1, \dots, g_n\}$. Then g_i , $i = 1, 2, \dots, n$ generates H_0 as a semigroup.

For every g_i take $a_i = g_i - 1$. Generate by the elements a_1, \dots, a_m a subalgebra U_0 in U . The subalgebra U_0 is nilpotent. It is easy to see that $g - 1 \in U_0$ for every $g \in H_0$ and $H_0 \subset 1 + U_0$. Take a subalgebra $U_0^* = \{U_0, 1\}$ in A . Here U_0 is the nilpotent ideal in U_0^* . The group $1 + U_0$ acts in U_0^* faithfully and unitriangularly. Hence, $1 + U_0$ is nilpotent and H_0 is also nilpotent. Thus, H is locally nilpotent.

Return to the situation when A is a PI -algebra and let

$$U_0 = 1 \subset U_1 \subset U_2 \subset A,$$

be the corresponding series of ideals. Take $H_1 = G \cap (1 + U_1)$ which is the kernel of the action G in A/U_1 . This group is locally nilpotent. In U_1 there is a directed system of the nilpotent ideals U_α of the algebra A . All $G \cap (1 + U_\alpha) = H_\alpha$ are the nilpotent normal subgroups in G and they constitute a directed system which generates H_1 .

Take, further, $G \cap (1 + U_2) = H_2$. This is a locally nilpotent normal subgroup in G which coincides the kernel of the action G in A/U_2 . The group H_1 is the kernel of action of the group H_2 in A/U_1 . This action is unitriangular and H_2/H_1 is a nilpotent group.

Consider a representation $G \rightarrow A/U_2$. It corresponds the faithful representation $G/H_2 \rightarrow A/U_2$. There is also an embedding $A/U_2 \rightarrow M_n(K)$. This implies the faithful embedding $G/H_2 \rightarrow \text{GL}_n(K)$. \square

Observe also the following. Let $K = \sum_{\alpha} P_{\alpha}$. For every P_{α} take an ideal U_{α} with $K/U_{\alpha} \simeq P_{\alpha}$ and $\bigcap_{\alpha} U_{\alpha} = 0$. For every α there is a homomorphism $\text{GL}_n(K) \rightarrow \text{GL}_n(P_{\alpha})$. Its kernel is the congruence-subgroup in $\text{GL}_n(K)$ modulo the ideal U_{α} . This leads to the presentation of $\text{GL}_n(K)$ as a subdirect product of the groups $\text{GL}_n(P_{\alpha})$.

4 Theorems 2 and 3

Let us repeat the formulations:

1. Every nil- PI -group is locally nilpotent -(Theorem 2).
2. In every PI -group the set of all nil-elements coincides with the locally nilpotent radical $\text{HP}(G)$ - (Theorem 3).

Theorem 2 follows from Theorem 3. Indeed, if every element g is a nil-element then $G = \text{HP}(G)$ and G is locally nilpotent.

As for Theorem 3 it is, in fact, proved in [Pla], [To1], [To2], [Pl3]. For the sake of the self-completeness of the text we will give here a proof of Theorem 3. Another reason is that the same scheme works for the proof of Theorem 6. We split the proof for 3 steps.

1. Let, first, G be a linear group, i.e., $G \subset \text{GL}_n(P)$. Check that every its nilpotent element lies in the radical $\text{HP}(G)$.

Let H be a subgroup in G generated by all its nil-elements. Show that H is locally solvable. Take in H a subgroup H_0 which is generated by a finite number of nil-elements. According to well-known theorem of A. Malcev [Ma] there is a system of normal subgroups T_{α} , $\alpha \in I$ in H_0 with the trivial intersection and with the finite quotients $H_{\alpha} = H_0/T_{\alpha}$. These quotients H_{α} are linear groups of the same dimension n over finite fields. Every H_{α} is generated by nil-elements and thus nilpotent by Baer's theorem. Therefore, H_{α} is solvable. Observe that all these H_{α} has the solvable length bounded by the number which depends only on n . Then H_0 is also solvable and H is locally solvable. It is known that locally solvable linear group is solvable [Su]. Thus, H is a solvable normal subgroup in G generated by nil-elements. According to [Pl1] such a group is locally nilpotent. All nil-elements of the group G lie in HP -radical of G .

2. Consider further the case $G \subset \text{GL}_n(K)$. The group G is approximated

by subgroups of linear groups $GL_n(P_\alpha)$. As before, let H be the subgroup in G generated by all nil-elements. This H is approximated by subgroups $H_\alpha \subset GL_n(P_\alpha)$. The subgroups H_α are generated by nil-elements and, hence, are solvable. The derived lengths are bounded for all H_α . Therefore, H is solvable. Since H is generated by nil-elements, then H is locally nilpotent invariant subgroup. Every nil-element lies in H , and therefore, in $HP(G)$.

3. General case. We have a chain

$$H_0 = 1 \subset H_1 \subset H_2 \subset H_3 \subset G,$$

where H_1 and H_2 are the same as in Proposition 3.1., and $H_3/H_2 = HP(G/H_2)$. Let g be a nil-element in G . Take a nil-element $\bar{g} = gH_2$ in G/H_2 . We have $\bar{g} \in H_3/H_2$, $g \in H_3$. Using again [P11], we have $g \in R(H_3)$. Since $R(H_3) \subset R(G)$, then $g \in R(G)$. \square

As we have mentioned, Theorem 2 is close to Procesi-Tokarenko theorem on periodic PI -groups. In some sense the theorem 2 is related also to the theorem from [WZ] where the profinite completion of a residually finite group is considered. The profinite setting also allows to proceed from nil-groups (not from Engel groups).

5 Generalizations

Let $u = u(x, y)$ denote the elements of the free group $F_2 = F(x, y)$. Consider a sequence $\vec{u} = u_1, u_2, u_3, \dots$. Such a sequence is called correct, if

1. $u_n(a, 1) = 1$ and $u_n(1, g) = 1$ for every n , every group G and every elements $a, g \in G$.
2. If $u_n(a, g) = 1$ then for every $m > n$ we have $u_m(a, g) = 1$ where a, g are the elements from G .

Thus, if the identity $u_n(x, y) \equiv 1$ is fulfilled in G then for every $m > n$ the identity $u_m(x, y) \equiv 1$ also holds in G .

For every correct sequence \vec{u} consider the class of groups $\Theta = \Theta(\vec{u})$ defined by the rule: $G \in \Theta$ if there exists n such that the identity $u_n(x, y) \equiv 1$ holds in G .

For every group G denote by $G(\vec{u})$ the subset in G defined by the rule: $g \in G(\vec{u})$, if for every $a \in G$ there exists $n = n(a, g)$ such that $u_n(a, g) = 1$. The elements of $G(\vec{u})$ are viewed as Engel elements in respect to the given correct sequence \vec{u} . We used to call these elements as \vec{u} -Engel-like elements.

If $\vec{u} = \vec{e} = e_1, \dots, e_n$ where the words $e_n(x, y)$ are defined by

$$e_1(x, y) = [x, y], \dots, e_n(x, y) = [e_{n-1}(x, y), y], \dots,$$

then $\Theta(\vec{e})$ is the class of all Engel groups. In case of finite groups the class $\Theta(\vec{e})$ coincides with the class of finite nilpotent groups.

For finite groups G the set $G(\vec{e})$ coincides with the nilpotent radical of the group G .

Problem 1. *Describe \vec{u} such that $\Theta(\vec{u})$ is the class of finite solvable groups.*

Concerning this problem see [BGGKPP], [Wi].

Problem 2. *Construct a sequence \vec{u} such that $G(\vec{u})$ is the solvable radical of every finite group G .*

It is not known whether there exist such \vec{u} . In both problems above emphasis is made on the fact that we are looking for two variable sequences. However, the similar problems can be considered also in the general case where the number of variables is not restricted.

In particular, we will consider some other approach to the problem of solvable radical description, which makes sense for finite groups too.

6 Further generalizations

For each given correct sequence \vec{u} define a new set $\vec{\vec{u}}$. Consider the free group $F = F(X, y)$ where $X = \{x_1, x_2, \dots, x_k, \dots\}$, and y is a distinguished variable. We will index words from F by the sequences of natural numbers (n_1, n_2, \dots, n_k) . Define the words

$$u_{(n_1, n_2, \dots, n_k)}(x_1, x_2, \dots, x_k; y)$$

by the rule: $u_{n_1}(x_1, y)$ coincides with the corresponding element of the sequence \vec{u} . Then, by induction,

$$u_{(n_1, n_2, \dots, n_k)}(x_1, x_2, \dots, x_k; y) = u_{n_k}(x_k; u_{(n_1, n_2, \dots, n_{k-1})}(x_1, x_2, \dots, x_{k-1}; y)).$$

The considered words obtained by superposition of two variable words.

It is easy to see that the following associativity takes place:

$$u_{(n_1, n_2, \dots, n_k)}(x_1, x_2, \dots, x_k; y) =$$

$$u_{(n_{l+1}, \dots, n_k)}(x_{l+1}, \dots, x_k; u_{(n_1, n_2, \dots, n_l)}(x_1, x_2, \dots, x_l; y)).$$

In particular,

$$\begin{aligned} u_{(n_1, n_2, \dots, n_k)}(x_1, x_2, \dots, x_k; y) = \\ u_{(n_2, \dots, n_k)}(x_2, \dots, x_k; u_{n_1}(x_1; y)) \end{aligned}$$

Correctness of the initial sequence \vec{u} induces some correctness of the system $\vec{\vec{u}}$.

For example, if for $l < k$ the group G satisfies the identity

$$u_{(n_l, \dots, n_k)}(x_l, \dots, x_k; y) \equiv 1,$$

or the identity

$$u_{(n_1, \dots, n_{l-1})}(x_1, \dots, x_{l-1}; y) \equiv 1,$$

then G satisfies the identity

$$u_{(n_1, \dots, n_k)}(x_1, \dots, x_k; y) \equiv 1.$$

There are also other relations of such kind.

For the given system $\vec{\vec{u}}$ consider the class of groups $\Theta = \Theta(\vec{\vec{u}})$. By definition, a group G belongs to Θ if an identity of the form

$$u_{(n_1, \dots, n_k)}(x_1, \dots, x_k; y) \equiv 1.$$

holds in G . From the observations above follows that the class Θ is a pseudovariety of groups. Besides that, for every group G we define a class of elements $G(\vec{\vec{u}})$ by the rule: $g \in G(\vec{\vec{u}})$ if for some $k = k(g)$ and for every sequence (a_1, \dots, a_k) of elements in G there is a set (n_1, \dots, n_k) such that

$$u_{(n_1, \dots, n_k)}(a_1, \dots, a_k; g) = 1.$$

is fulfilled. Here, the set (n_1, \dots, n_k) should be compatible with the set (a_1, \dots, a_k) . This means that n_1 depends on a_1 and g , and does not depend on (a_2, \dots, a_k) ; n_2 depends on a_1, a_2 , and g , and does not depend on (a_3, \dots, a_k) , etc.; n_k depends on a_1, a_2, \dots, a_k and g .

Here arises a general problem of some description of the sets $G(\vec{\vec{u}})$ for the different $\vec{\vec{u}}$.

Consider a special case of $G(\vec{\vec{u}})$. Denote $\varepsilon = \vec{e}$ and take the sequence

$$\varepsilon : e_1, e_2, \dots, e_n, \dots$$

Consider the system $\overrightarrow{\varepsilon}$ and using this system define *quasi-nil elements* in groups. An element $g \in G$ is called *quasi-nil* if $g \in G(\overrightarrow{\varepsilon})$. This means that for g there is $k = k(g)$, such that for any sequence a_1, \dots, a_k , $a_i \in G$, there is a compatible set (n_1, \dots, n_k) such that

$$\varepsilon_{(n_1, \dots, n_k)}(a_1, \dots, a_k; g) = 1.$$

For the sequence ε we have also the class of groups $\Theta(\overrightarrow{\varepsilon})$. The groups from this class can be considered simultaneously as generalized nilpotent and generalized solvable groups.

Denote by $E_{(n_1, \dots, n_k)}$ the variety defined by the identity

$$\varepsilon_{(n_1, \dots, n_k)}(x_1, \dots, x_k; y) \equiv 1.$$

The class Θ is the union of such varieties. The variety of the type $E_{(1, \dots, 1)}$ is the nilpotent variety, while the variety of the type $E_{(1, 2, \dots, 2)}$ contains the solvable subvariety. Besides that a product of varieties of the type $E_{(n_1, \dots, n_k)}$ is a subvariety in the variety of the same type. This observation relates, in particular, to the product $E_{n_1} E_{n_2} \cdots E_{n_k}$. This variety lies in the variety $E_{(n_k, n_{k-1}+1, \dots, n_1+1)}$.

Return now to quasi-nil elements in groups. Let $k = k(g)$ be the minimal number such that for every (a_1, \dots, a_k) there is a compatible set (n_1, \dots, n_k) with

$$\varepsilon_{(n_1, \dots, n_k)}(a_1, \dots, a_k; g) = 1.$$

We call such $k = k(g)$ the nil-order of g . Nil-order 1 means that the element is a nil-element, nil-order 2 means that the element is not nil, but for a_1 and a_2 there are n_1, n_2 with $\varepsilon_{(n_1, n_2)}(a_1, a_2; g) = 1$. In general for $k-1$ we have some elements $(a_1^0, \dots, a_{k-1}^0)$ such that

$$\varepsilon_{(n_1, \dots, n_{k-1})}(a_1^0, \dots, a_{k-1}^0; g) \neq 1.$$

for the arbitrary compatible (n_1, \dots, n_{k-1}) .

Let us add to $(a_1^0, \dots, a_{k-1}^0)$ an arbitrary element a . Then for the sequence $(a_1^0, \dots, a_{k-1}^0, a)$ there is a corresponding set $(n_1^0, \dots, n_{k-1}^0, n)$ with the condition

$$e_n(a; g_0) = 1,$$

where

$$g_0 = \varepsilon_{(n_1^0, \dots, n_{k-1}^0)}(a_1^0, \dots, a_{k-1}^0; g).$$

Here, the element g_0 is not trivial, the element a does not depend on g_0 . The equality $e_n(a, g_0) = 1$ now means that the element g_0 is a non-trivial nil-element.

Simultaneously, we proved the following

Proposition 6.1. *If a group G contains a non-trivial quasi-nil element g then G contains also a non-trivial nil-element g_0 .*

Note now the next two properties related to the definition of the quasi-nil element.

1. Let H be a subgroup in G and $g \in H$ be a quasi-nil in G . Then g is quasi-nil element in H .
2. Let a surjection $\mu : G \rightarrow H$ be given and let g be a quasi-nil element in G . Then $\mu(g)$ is a quasi-nil element in H .

Indeed, take $k = k(g)$ and the corresponding presentation

$$\varepsilon_{(n_1, \dots, n_k)}(a_1, \dots, a_k; g) = 1.$$

Then

$$\varepsilon_{(n_1, \dots, n_k)}(\mu(a_1), \dots, \mu(a_k); \mu(g)) = 1.$$

Here, $\mu(a_1), \dots, \mu(a_n)$ are arbitrary elements in H .

It is clear that along with quasi-nil elements it is quite natural to define quasi-Engel elements which generalize Engel elements.

7 Some radicals

Let G be a group. Consider in G the locally nilpotent radical $HP(G) = R(G)$ and the locally noetherian radical $NR(G)$. The corresponding upper radicals will be denoted by $\widetilde{R}(G)$ and $\widetilde{NR}(G)$. These radicals are obtained by iterations of the initial $R(G)$ and $NR(G)$. Namely, consider the series (upper radical series)

$$1 = R_0 \subset R = R_1 \subset \dots \subset R_\alpha \subset \dots,$$

where $R_{\alpha+1}/R_\alpha$ is $R(G/R_\alpha)$. Such a series terminates at some $R_\gamma = \widetilde{R}(G)$. Then $\widetilde{R}(G)$ is the upper radical for the radical $R(G)$. The factor group $G/\widetilde{R}(G)$ is locally nilpotent semi-simple. i.e., it does not contain non-trivial locally nilpotent normal subgroups.

The radical \widetilde{R} is defined also by the class of groups G which has ascending normal series with locally nilpotent factors. Such groups are called radical groups (see [P11]). In finite groups the radical $\widetilde{R}(G)$ coincides with the solvable radical of a group.

The radical $\widetilde{NR}(G)$ is defined following the same scheme as for the radical $\widetilde{R}(G)$. If $\widetilde{NR}(G) = G$, the group G is called noetherian radical group.

8 Theorems on radical characterization

Let us take in the upper radical series of a group G the members with finite indexes

$$1 = R_0 \subset R_1 \subset \dots R_k \subset \dots,$$

Proposition 8.1. *An element g which belongs R_k for some k and does not belong to R_{k-1} is a quasi-nil element of the nil-order k .*

Proof. For the case $g \in R_1$ this is true. Further we use induction. Suppose that for $g \in R_{k-1}$ it is proved that the nil-order of this g is $\leq k-1$. Let $g \in R_k$. Take a sequence of elements a_1, \dots, a_k in G and for a_1 and g find n_1 with $e_{n_1}(a_1, g) \in R_{k-1}$. Apply induction to the element $e_{n_1}(a, g)$. We have:

$$\varepsilon_{(n_2, \dots, n_k)}(a_2, \dots, a_k; e_{n_1}(a_1, g)) = 1 = \varepsilon_{(n_1, \dots, n_k)}(a_1, \dots, a_k; g).$$

Hence, the nil-order of the element g is $\leq k$. Prove further that it is exactly k . Let g is of the order $l \leq k$. Take $a_1, \dots, a_l, n_1, \dots, n_l$ such that

$$\varepsilon_{(n_1, \dots, n_l)}(a_1, \dots, a_l; g) = 1 = e_{n_l}(a_l; \varepsilon_{(n_1, \dots, n_{l-1})}(a_1, \dots, a_{l-1}; g)).$$

The element a_l does not depend on $g_0 = \varepsilon_{(n_1, \dots, n_{l-1})}(a_1, \dots, a_{l-1}; g)$, and all these g_0 are nil-elements (for all a_1, \dots, a_{l-1}). Some of g_0 are non-trivial and all of them lie in R_1 . Consider G/R_1 . Here all \bar{g}_0 are trivial and the nil-order of \bar{g} is $\leq l-1$. By the assumption of induction $\bar{g} \in R_l/R_1$. Then $g \in R_l$. By the condition g does not belong to R_{k-1} . Then $l = k$. \square

Proposition 8.2. *Let $\widetilde{NR}(G) = G$ and $\widetilde{R}(G)$ be the radical. Then every quasi-nil element $g \in G$ belongs to $\widetilde{R}(G)$.*

Proof. Let g be a quasi-nil element which does not belong to $\widetilde{R}(G)$. Element $\bar{g} = g\widetilde{R}(G)$ is quasi-nil in the semi-simple group $\bar{G} = G/\widetilde{R}(G)$. If $g \neq 1$ then there exists a non-trivial nil element in \bar{G} . We came o contradiction with the semi-simplicity of G . \square

Theorem 4. *Let $\widetilde{NR}(G) = G$ and let the upper radical series in G has a finite length. Then $\widetilde{R}(G)$ coincides with the set of all quasi-nil elements in G .*

Proof. The proof follows from Proposition 1 and Proposition 2.

We have seen that every quasi-nil element in $\widetilde{NR}(G) = G$ lies in $\widetilde{R}(G)$ for upper radical series of any length. However, in this general situation we cannot state that every element from $\widetilde{R}(G)$ is quasi-nil. In order to include this case in the general setting we define unbounded quasi-nil elements. In this unbounded approach we do not fix $k = k(g)$, since we do not know in advance what are the length of words which are related with the given g . Thus we consider infinite sequences $\bar{a} = (a_1, a_2, \dots, a_k, \dots)$. We call an element unbounded quasi-nil, if for any \bar{a} there are $k = k(\bar{a}, g)$ and compatible (n_1, n_2, \dots, n_k) such that

$$\varepsilon_{(n_1, n_2, \dots, n_k)}(a_1, a_2, \dots, a_k; g) = 1.$$

Theorem 5. *For any group G every element in $\widetilde{R}(G)$ is an unbounded quasi-nil element.*

Proof. We start from the upper radical series

$$1 = R_0 \subset R_1 \subset R_2 \subset \dots \subset R_\alpha \subset \dots \subset R_\gamma = R$$

and use the induction. For $g \in R_1$ the statement is evident and let for all $\beta < \alpha$ the statement is true. Show that every $g \in R_\alpha$ is unbounded quasi-nil.

If α is terminal then $g \in R_\beta$ with $\beta < \alpha$ and g is unbounded quasi-nil.

Suppose now that there exists $\alpha - 1$. For the given $g \in R_\alpha$ take a sequence

$$\bar{a} = (a_1, a_2, \dots, a_k, \dots).$$

For a_1 we find n_1 with $e_{n_1}(a_1, g) \in R_{\alpha-1}$. The element $e_{n_1}(a_1, g)$ is unbounded quasi-nil. For this element take the sequence a_2, \dots, a_k, \dots , and let (n_2, \dots, n_k, \dots) be defined for this sequence. Here, k depends also on a_1 . We have

$$\begin{aligned} \varepsilon_{(n_2, \dots, n_k)}(a_2, \dots, a_k; e_{n_1}(a_1, g)) &= 1 \\ &= \varepsilon_{(n_1, n_2, \dots, n_k)}(a_1, a_2, \dots, a_k; g). \end{aligned}$$

Thus, the element g satisfies the condition to be unbounded quasi-nil.

9 Again about PI-groups

Theorem 3 can be applied to finite groups. It can be also applied to linear groups over fields and, as we will see soon, to any *PI*-groups. In these cases the conditions of the type $\widetilde{NR}(G) = G$ are not necessary.

Theorem 6. *If G is a *PI*-group, then its "solvable" radical $\widetilde{R}(G)$ coincides with the set of all quasi-nil elements.*

The proof of this theorem follows the scheme used in the proof of Theorem 2. The only observation has to be taken into account is the fact that in every solvable group its solvable radical coincides with the set of its nil-elements. \square

In particular, we can state that a *PI*-group is "solvable" (in the sense that $\widetilde{R}(G) = G$) if and only if all elements in G are quasi-nil.

From the theorem 2 follows that if in *PI*-group G every two elements generate a nilpotent subgroup then the whole group is locally nilpotent. Now we consider the case when every two elements generate a solvable subgroup.

Theorem 7. *Let G be a *PI*-group and let every two elements in G generate a solvable subgroup. Then G is solvable modulo locally nilpotent radical $HP(G)$.*

Proof. It is sufficient to consider the case when G is a subgroup in a $GL_n(K)$, where K is a direct sum of fields. If K is a field the proof follows from [Th], [Su]. The proof for the general case imitates the reduction to the field case in the previous theorem. \square

Remark 1. In every *PI*-group G the group $\widetilde{R}(G)/R(G)$ is solvable. However, it is not clear whether the group $\widetilde{R}(G)$ is always locally solvable.

10 Conclusion

All above can be applied to finite groups. However, the problems 1 and 2 remains open. Their solution should use the "subtle" theory of finite groups. This is the classification of finite simple groups and their automorphisms, equations in finite simple groups, etc. Here some algebraic geometry can be used. Besides that, along with Engel-like elements the corresponding Engel-like automorphisms should be considered.

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